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The Clock Paradox as a Cosmological Problem

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Abstract

In this paper we discuss the clock paradox within the framework of the general theory of relativity. It is shown that in general *the aging asymmetry* exists. We also argue that the clock paradox, according to Mach's principle, is essentially a cosmological problem.

1. *Introduction*

The problem of the clock paradox in the relativity theory is more than sixty years old. Perhaps no other problem in relativity has had so many papers written about it. There is no need for us to review the historical development of this problem here. Some remarks, however, must be made. It seems that the majority of physicists (Tolman, 1934; Arzelies 1966; Møller, 1967), including Einstein (1918) himself, tend to agree that the *aging asymmetry* in the concerned problem does exist. Most of the authors have, however, restricted their discussions to the special theory of relativity (STR) or at most to a Newtonian approximation about the time retardation based on the general theory of relativity (GTR). Such a discussion, of course, is ambiguous and incomplete. Recently, Wu and Lee (1972) have intended to resolve this problem on a more general footing. But, as pointed out by Sachs (1973), their conclusion, based on a certain approximation, also is ambiguous. This approximation actually still limits the problem within the framework of STR.

Sachs himself has tried to resolve the clock paradox by a general argument from the view of GTR (1971). His result is rather astonishing: There is no *aging asymmetry* at all! This conclusion has caused a great dispute from many physicists (Terrell *et al.*, 1972). Actually, Sachs holds a different view toward the fundamental field quantities $g_{\alpha\beta}$ from the conventional GTR (Sachs 1967, 1968, 1970). He argues that it follows from the features of the transformation group that underlies GTR, that the symmetric-tensor representation cannot be the most general representation of this theory. This is because the invariance group (the Einstein group) is a 16-parameter Lie group and the basic equations that describe the metrical field necessarily entail 16 (rather than 10) relations at each space-time point. The fundamental reason for the reduction from 16 to 10 relations in GTR is the further symmetry that is imposed on the formalism regarding covariance with respect to space or time reflections. Since the principle of general relativity refers to frames of reference that are in relative *motion*—a presumably continuous entity—the reflection symmetry elements in the invariance group are an undue restriction. With such an opinion, Sachs noticed that the variables that solve the underlying field equations must be defined, most primitively, in terms of a set of four quarternion fields, $q_\alpha(x)$, which behave geometrically as a four-vector in a Riemannian space, but behave algebraically as the quarternions. Since each of the quarternions is a Hermitian 2-dimensional matrix and thereby depends on four real fields in their matrix components, it follows that the quarternion field equations are in terms of $4 \times 4 = 16$ relations at each x, in accordance with the previous argument. In Sachs' theory, in order to incorporate properly the algebraic properties of the quarternion into the metric of a Riemannian Space, the differential interval in four-space is defined as

$$ds = q_\alpha(x)\, dx^\alpha \tag{1.1}$$

When this is multiplied by its conjugate quarternion $\widetilde{ds}$, one obtains a real number which corresponds to the quadratic form of the line element in the conventional GTR

$$ds\, \widetilde{ds} = -\tfrac{1}{2}(q_\alpha \tilde{q}_\beta + q_\beta \tilde{q}_\alpha)\, dx^\alpha\, dx^\beta \Rightarrow g_{\alpha\beta}\, dx^\alpha\, dx^\beta \tag{1.2}$$

The formula (1.1) and its conjugate can be recognized as the factorization of the squared interval in GTR. This is precisely the same as that the Klein-Gordon equation in STR can be factorized into a pair of conjugated two-component spinor equations, which in turn leads to the conventional form of the Dirac equation. Regardless of what the theory implies, it at least provides a way to factorize the squared interval in GTR. Sachs has used the factorized ds (or $\widetilde{ds}$) to discuss the clock paradox. He considers a closed path line integral $\oint ds = \oint q_\alpha\, dx^\alpha$ in four-space, and shows that it must vanish. This indicates no *aging asymmetry* in the problem of clock paradox; i.e., there is no paradox at all. To reach this result, Sachs believes that given appropriate boundary conditions q_α, which are solvable from the field equations, must be analytic at every space-time point.

In this paper, we will also adopt the quarternion representation of the differential interval as a factorization of the squared interval in GTR. We, however, contend that the consideration of the closed path line integral in Sachs' argument is not rigorous. The main point is that a closed path line integral for *resolving* the clock paradox (as Sachs claimed) cannot be arbitrarily drawn. We believe that a closed path line integral with physical significance in the problem of the clock paradox exists only in the space-time with an intrinsic property of rotation (see Section 2). This immediately yields $\oint ds \neq 0$ from the quarternion representation of the differential interval; i.e., there is aging symmetry. We further argue that the clock paradox is essentially a cosmological problem according to Mach's principle. From such a view the problem becomes so fundamental that it is not strange why people have been puzzled for such a long period.

2. *The Significance of* $\oint ds$

First, we must make it clear what the significance of a closed path line integral in the concerned question is. Let us recall the problem of the clock paradox: Consider a pair of identical clocks, which initially have no relative motion at the same point, projected into two different trajectories to meet finally at another point without relative motion in the space-time. Will the two time intervals elapsed be different? Will they both agree exactly on that result? The pair of identical clocks can be two identical twins, or two identical decaying particles, or two identical atomic clocks, etc. Anyhow, they must be physical entities that can show the property of time flowing. Hence the closed path line integral in the space-time cannot be fictitious. It must move along with the physical entity. And when we move along with the physical entity to complete a closed path, the space-time, in which the physical entity is embedded, must contain an intrinsic property of rotation. In Section 4 we will be more explicit in our words in describing an artificial arrangement of the line integral.

The situation is more analogous to the case of irrotational and rotational flow in fluid mechanics. The flow can be irrotational when we consider a *fictitious* closed contour and find the associated vorticity vanished. But if we want to move along with a fluid element and complete a *closed* contour, the associated flow is necessarily rotational. We therefore assert that in the discussion of the clock paradox a closed path line integral $\oint ds$ in four-space is physically significant only when the ambient space-time possesses the intrinsic rotational character. We also have to emphasize that while the rotation of the ambient space-time follows from the existence of $\oint ds$, it does not imply that the existence of $\oint ds$ follows from the existence of rotation.

3. *The Rotating Space-Time*

The rotation mentioned in the previous section is an intrinsic character of the space-time. In GTR it must be explicitly stored in the expression of the fundamental field quantities—the metric tensor $g_{\alpha\beta}$. To simplify the discussion, we choose a reference frame such that the squared interval can be written as

$$ds^2 = c^2\, dt^2 + g_{ij}\, dx^i\, dx^j + g_{0i}\, dx^0\, dx^i + g_{i0}\, dx^i\, dx^0 \qquad (3.1)$$

which the Latin indices run from 1 to 3. This reference frame, in which $g_{00} = 1$, certainly can be found from the coordinate transformation. To visualize the intrinsic character of rotation, we further choose the so-called comoving reference frame, i.e. in such a frame the trajectories of the rest matter are geodesics in the four-space. The geodesic equations are

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = 0 \qquad (3.2)$$

or written in terms of the four-velocity $u^\alpha = dx^\alpha/ds$,

$$\frac{du^\alpha}{ds} + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma = 0 \tag{3.3}$$

The four-velocity of the rest matter is $u^\alpha = (c, 0, 0, 0)$. Substituting this in equation (3.3) and from $g_{00} = 1$, we find

$$g_{0k,\,0} = 0 \tag{3.4}$$

So $g_{00} = 1$, and $g_{0k,\,0} = 0$ are the conditions for a reference frame to be comoving. This includes four conditions, and certainly can always be fixed through four coordinate transformations $x^\alpha \to x'^\alpha$.

In the comoving reference frame, the rotational character can be easily recognized from the metric tensors $g_{\alpha\beta}$. In fluid mechanics, the rotational character is determined from the curl of the velocity field $\nabla \times \mathbf{v}$—the vorticity. In the geometrical field of the four-space, we can construct a corresponding four-vector, or four-vorticity

$$\Omega^\alpha = \frac{\epsilon^{\alpha\mu\nu\gamma}}{\sqrt{-g}} \{u_\mu u_{\nu,\,\gamma}\} = -e^{\alpha\mu\nu\gamma}\{u_\mu u_{\nu,\,\gamma}\} \tag{3.5}$$

where $e^{\alpha\mu\nu\gamma} = -\epsilon^{\alpha\mu\nu\gamma}/\sqrt{-g}$, g is the determinant of $g_{\alpha\beta}$,

$$\epsilon_{\alpha\mu\nu\gamma} = \begin{cases} 0, & \text{if any two indices are equal.} \\ \pm 1, & \text{according to } (\alpha\mu\nu\gamma) \text{ being an even or odd permutation of } (0, 1, 2, 3). \end{cases}$$

$$\{u_\mu u_{\nu,\gamma}\} = \frac{1}{3!}(u_\mu(u_{\nu,\gamma} - u_{\gamma,\nu}) + u_\nu(u_{\gamma,\mu} - u_{\mu,\gamma}) + u_\gamma(u_{\mu,\nu} - u_{\nu,\mu})),$$

$$u_{\nu,\gamma} = \frac{\partial u_\nu}{\partial x^\gamma}, \text{ etc.}$$

In the comoving system, $u_\alpha = c(1, g_{01}, g_{02}, g_{03})$. Equation (3.5) is equivalent to

$$\Omega^\alpha = -e^{\alpha\mu\nu\gamma}\{g_{0\mu} g_{0\nu,\gamma}\}, \text{ with } g_{00} = 1 \tag{3.6}$$

If any component of Ω^α is not zero, the space-time obviously possesses the intrinsic rotational character. In fact, this rotational character can be understood in better perspective when we study the motion of a test particle. However, we do not bother ourselves with this problem here.

From equation (3.6), we reach the conclusion: *For all $i \neq j$, if the equality*

$$g_{0i,j} = g_{0j,i} \tag{3.7}$$

holds, then the corresponding space-time is not rotating, otherwise, it is rotating.

4. *The Proof of $\oint ds \neq 0$*

The motion of a *free* physical entity in a four-space is determined by the principle of least action

$$\delta \int ds = 0 \tag{4.1}$$

From this we obtain the geodesic equations (3.2) in GTR. If a physical entity starts from point 1 to point 2 in the four-space, equation (4.1) then gives only a line integral

$$_g\int_1^2 ds \tag{4.2}$$

where the subindex g represents the geodesic line.

However, for a pair of identical clocks, we actually consider two different paths from the point 1 to 2 in the four-space. This must correspond to two different geodesic lines, and hence, correspond to two different four-spaces. How can we consider these two four-spaces as a space-time with rotation? We can make such an artificial consideration: Consider the pair of clocks identified as A and B. The clock A moves along the geodesic line g_1 and B along g_2. Let us look at A and neglect B first. We consider A moves along the reversed geodesic line $-g_1$ from the point 2 to 1. This corresponds to a space-time which is the time reversal of the original one. When the point 1 is reached, we begin to look at B (≡A actually at this point) and forget A. The clock B moves along the geodesic line g_2 and finally arrives at the point 2. Since this is a closed path, the embedding space-time must be rotating (i.e. equation (3.6) does not hold completely in the space-time region that we are interested).

Since $ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta$, so $\oint ds = \oint (g_{\alpha\beta}\, dx^\alpha\, dx^\beta)^{1/2}$. There is no general way to get rid of the square root in the integrand of the closed path line integral. We have seen, however, in the Introduction that the product of the quarternion representation of the differential interval and its conjugate in Sachs' theory is one-one correspondent to the squared interval in GTR. So it is natural to consider ds and $\widetilde{ds}$ as factors of ds^2. Instead of manipulating $\oint ds = \oint (g_{\alpha\beta}\, dx^\alpha\, dx^\beta)^{1/2}$, we can therefore calculate $\oint ds = \oint q_\alpha\, dx^\alpha$. For this we follow the same line as Sachs'. But we hold a different point of view. Sachs uses the quarternion representation as the basis of his relativity theory. We use it only because of its advantage in factorizing the squared interval ds^2 in GTR.

Now, consider $\oint q_\alpha\, dx^\alpha$. It is trivial that this closed path line integral in the four-space vanishes if (and only if) $q_\alpha\, dx^\alpha$ is an exact differential. And this is the case if (and only if)

$$q_{\alpha,\beta} = q_{\beta,\alpha} \tag{4.3}$$

everywhere in the interested region. This is an elementary theorem in calculus. In the following, we prove that equations (3.7) and (4.3) are equivalent in the comoving reference frame.

In the comoving reference frame, $g_{00} = 1$, so q_0 must be the unit two-dimensional matrix. Further, since $g_{\alpha\beta} \Rightarrow -\frac{1}{2}(q_\alpha \tilde{q}_\beta + q_\beta \tilde{q}_\alpha)$, the condition $g_{0k,0} = 0$ for the comoving reference frame is equivalent to $q_{k,0} = 0$. From this information it is easy to see that equation (3.7) is equivalent to $q_{i,k} = q_{k,i}$.

This, together with the fact that q_0 is the unit two-dimensional matrix, yields the result that equations (3.7) and (4.3) are equivalent.

Because the closed path line integral exists only in a rotating space-time as we have emphasized in Section 2, we conclude that $\oint ds \neq 0$.

Our proof of $\oint ds \neq 0$, although based on the choice of the comoving reference frame, is generally true, of course, since ds is a Lorentz invariant. The sign of $\oint ds$ which is determined from the sense of the rotation in turn determines which clock (A or B) is older.

5. *Mach's Principle and Cosmology*

Sachs has made the following statements in his argument about the *illogic of asymmetric aging*: Some internal physical processes within a physical entity are responsible for the physical effect we call 'aging'. Relativity theory does not claim that any effect of these physical processes on aging are in any way affected by the relative motion of some observers who may be looking at the considered system from a moving platform. To accept the *aging asymmetry* means a regression to the classical view in which time is indeed an absolute measure with its own physical manifestations.

These statements may not be true if Mach's principle is incorporated properly in GTR. Previously, we verify the *aging asymmetry* by considering the rotation of a space-time. But how can a space-time rotate? With respect to what does it rotate? These questions on a noninertial reference frame are very old—older than the clock paradox. One must retrace back to Newton's time when his contemporary, Leibnitz, argued with him about the absolute reference frame. The nineteenth century philosopher Mach has tried to resolve this dispute from a relativistic view and reached the conclusion: The inertial property of a material body is determined by the distribution of the matter in the universe. Einstein took Mach's principle seriously, and it was his hope that this principle could somehow be incorporated into GTR. In cosmology, however, there is the so-called Gödel solution (Godel, 1949) which satisfies the Einstein field equations but violates Mach's principle. This gives an indication that Mach's principle is not *completely* incorporated into GTR. The comoving matter in a Gödel universe undergoes an intrinsic uniform rotation. The same questions then arise: How can the universe rotate? With respect to what does it rotate? Some physicists (e.g., Wheeler, 1964) regard that Mach's principle must be incorporated into GTR by imposing appropriate boundary conditions. This makes Mach's principle become a filter in choosing the cosmological model. Some other physicists take different views. For example, Dicke and Brans consider that Mach's principle can be incorporated into GTR through the introduction of a long-range scalar field. This is the theoretical basis of the Brans-Dicke theory (1961).

The rotational character in the clock paradox, however, differs from that in the Gödel universe, in which the intrinsic rotation represents a bulk rotation of the whole universe. The rotation in our discussion of the clock paradox must be *local*. It represents a *local* rotation with respect to the distant stars and the whole universe. According to Mach's principle a certain kind of interaction must exist between the matter of the whole universe and the physical entity in the local region. This interaction can be transmitted through the imposition of appropriate boundary conditions or through the scalar field in the Brans-Dicke theory. Hence, cosmological interaction also comes onto the stage and is responsible for the 'aging' process. This contradicts with what Sachs says that only internal physical processes are responsible for the 'aging' process. To accept the *aging asymmetry* does not mean a regression to the absolute time.

We remark that the problem of the clock paradox is essentially a cosmological problem. Only when the interaction between the considered physical entities and the mass-energy distribution of the whole universe is explicitly expressed can the clock paradox be completely solved. In this view, the problem becomes so fundamental that it is no wonder why people have long been puzzled.

6. *Discussions*

We shall cast more doubts on Sachs' mathematical proof of $\oint ds = 0$. Sachs had restricted his discussion in the two-dimensional differential interval $ds = q_1\, dx^1 + q_2\, dx^2$. He claims that this interval can be written as $Re f(z)\, dz$ where $f(z)$ is a single-valued function of $z = x^1 + ix^2$ that is analytic at all points in and on the closed curve of integration. This automatically warrants $\oint ds = 0$. Such an argument is not rigorous for the following reasons: (a) The condition for the existence of a function like $f(z)$ is that each of the four real fields that make up the quarternion variables must be separately single valued and analytic. This is actually condition (3.7). It is easy to see from our discussion that this does not hold in general. (b) It seems to us that Sachs did not consider the cosmological effect into boundary conditions. The analylicity of the function $f(z)$ then actually can be extended to infinity in space. Doesn't it imply that $f(z)$ is a constant function? These two difficulties have been eliminated in this paper.

Equation (3.7) is a criterion to discover the intrinsic character of rotation. When it is not satisfied one has the necessary, but insufficient, condition to complete a closed path line integral. The sufficient condition requires a more thorough consideration. However, the necessary condition is *sufficient* in proving the *aging asymmetry*.

Our result is consistent with the majority of conclusions from the consideration of STR. This corresponds to the circumstance that the noninertial effect, or the cosmological interaction, is weak. In this limit one can treat the problem as coordinate transformations between inertial reference frames in the spirit of STR. When the noninertial effect, or the cosmological interaction, is strong, the problem becomes cosmological. The problem calls a proper incorporation of Mach's principle into GTR.

Philosophers may think that the *aging asymmetry* is hardly understandable. We think this is not true. Let us make a philosophical remark: A *great* life lives longer because of its stronger *cosmological* sense.

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